Estimation of VaR bounds under dependence uncertainty and their use for the SCR calculation in Solvency II

Stanisław Wanat¹, Krzysztof Guzik²

Abstract
The subject of the article is part of the discussion regarding the VaR estimation for aggregated risk under conditions of uncertain structure of dependence, i.e. VaR for the sum of individual risk with known marginal distributions and unspecified dependence structure. It demonstrates that the use of a standard formula in accordance with Article 115 of the Delegated Regulation (EU) 2015/35, without identifying the actual dependency structure, may lead to an erroneous estimation of solvency capital requirements for non-life premium and reserve risk. It also indicates how large the errors may be. The analysis was carried out using the dependence uncertainty spread estimation methods known in the literature.

Keywords: dependence structure, Solvency Capital Requirements, risk aggregation, VaR bounds.

JEL Classification: C150, C580, G220

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1 Introduction
The Value-at-Risk (VaR) has been mainly used for measuring risk at banks and insurance companies for the last three decades. This measure constitutes basis for making decisions on limiting risk exposure to an acceptable level, depending on available economic capital, and it is also used for determining regulatory capital which are to guarantee the solvency. For example, VaR is used in Solvency II in the standard method of determining the solvency capital requirements (SCR). So it is not surprising that in the subject literature there have been and are still wide-ranging debates being conducted on issues related to diversification, aggregation, dependence uncertainty, economical interpretation, optimization, extreme behavior, robustness, and back-testing of VaR; see, among others, Embrechts et al. (2015), Embrechts et al. (2014) and Emmer et al. (2014) for more details.

The article subject fits into the discussion on assessing VaR for the aggregated risk in the conditions of the unspecified dependence structure, that is VaR for the sum of individual risks with known marginal distributions but the unknown dependence among them. It broadens the existing subject literature in a scope of quantitative, multivariate modeling of the risk in the

¹ Corresponding author: Cracow University of Economics, Department of Mathematics, 27 Rakowicka St. 31-510 Cracow, Poland, wanats@uek.krakow.pl
² Cracow University of Economics, Department of Mathematics, 27 Rakowicka St. 31-510 Cracow, Poland, guzikk@uek.krakow.pl.
process of determining the SCRs. The work shows that the use of a standard formula in accordance with Article 115 of the Delegated Regulation (EU) 2015/35, without identifying the actual dependency structure, may lead to an erroneous estimation of solvency capital requirements for non-life premium and reserve risk. It also indicates how large the errors may be. The analysis was carried out using the dependence uncertainty spread estimation methods known in the literature. To the authors’ best knowledge, the results of that type of studies have not been presented in the literature yet.

The other part of the article is organized in the following way. In the next chapter we present the methods of assessing the limitations for VaR (VaR bounds). In chapter 3 we present briefly the standard method assessing the solvency capital requirements for the non-life premium and reserve risk used in Solvency II. Then in chapter 4 we discuss the assumptions and the results of the simulation study performed, and in the last fifth chapter we present conclusions.

2 Methods of assessing the bounds for VaR – a literature review

We consider a portfolio consisting of $n$ random variables $L_1, L_2, ..., L_n$ (individual risks associated with a given business line or a risk type) with the finite mean and the variance. We assume marginal distributions $F_j (j = 1, ..., n)$ corresponding to them are provided. For random variable $L = \sum_{j=1}^{n} L_j$, modelling the aggregate portfolio loss, at the established confidence level $1-\alpha$, we define Value-at-Risk (VaR):

$$VaR_{1-\alpha}(L) = \inf \{x \in R : F_L(x) \geq 1 - \alpha \},$$

where $F_L$ is the distribution of $L$. The Value-at-Risk depends only on the dependence structure of the multivariate random variable $(L_1, ..., L_n)$, and the copula $C$ describes this structure on basis of the Sklar’s theorem. When we know the copula $C$ we are able to determine VaR for the random variable $L$ in a simple way, otherwise we can give only its bounds.

The first results for assessing VaR in the case of the unspecified dependence structure for two random variables are presented in the works (Makarov, 1981) and (Frank et al., 1987) and independently in (Rüschendorf, 1982). This issue has been discussed recently in many works (Puccetti and Rüschendorf, 2012a, 2012b, 2014; Embrechts et al., 2013; Puccetti et al., 2013; Bernard et al., 2016, 2017). It should be pointed out that the results obtained for three or more random variables require certain assumptions such as e.g. identically distributed risks having monotone densities (Puccetti, 2013; Puccetti and Rüschendorf, 2013). For the inhomogeneous case there are no analytical solutions enabling to determine „reasonable” VaR bounds. The
studies on assessing VaR resulted in proposing numerical algorithms which do not require the assumptions mentioned above. We present briefly three of them.

Rearrangement Algorithm (RA) was proposed by Puccetti and Rüschendorf (2012b) and Embrechts et al. (2013). On its basis the VaR assessments can be determined at known, not necessarily homogeneous, marginal distributions. In the RA algorithm the Value-at-Risk measure such as Expected Shortfall (ES) was used, ES unlike VaR meets the property of sub-additivity and is thus coherent:

$$ES_{1-\alpha}(L) = \frac{1}{\alpha} \int_{1-\alpha}^{1} VaR_q(L) dq .$$

Value $ES_{1-\alpha}(L)$ is the average loss of all upper VaRs from level $1 - \alpha$, when those losses at the established confidence level $1 - \alpha$ exceed $VaR_{1-\alpha}(L)$. Therefore, the upper estimate of VaR by this measure is following:

$$ES_{1-\alpha}(L) \leq \overline{ES}_{1-\alpha}(L) = \sum_{j=1}^{n} ES_{1-\alpha}(L_j).$$

Since it results from the ES definition that $\overline{VaR}_{1-\alpha}(L) \leq \overline{ES}_{1-\alpha}(L)$ is the bounds for VaR we obtain by estimating ES. In the RA algorithms at the established number of discretization points N from the tail of distribution $[1 - \alpha, 1]$ for each random variable $L_j$ we determine $N$ of realization of this variable by making matrix $X$ of dimension $N \times n$. Then, the value of each random variable we permute countermononic in relation to other variables so that realizations of the sum of those random variables in the $N$ cases were similar to each other. In this iterative way we obtain the sought estimate at the fixed level absolute tolerance $\epsilon$.

In turn, Adaptive Rearrangement Algorithm was an answer to weak efficiency of the RA algorithm at a greater number of the random variables. It was proposed in work (Hofert et al., 2017). The authors have considered in it at what level values $N$ and $\epsilon$ should be established for the algorithm to be effective (the RA algorithm authors do not take it into consideration). A course of the ARA algorithm is analogous to RA one, with the difference that a number of $N$ discretization points is selected depending on the algorithm convergence degree. Additionally, in contrast to the absolute tolerance $\epsilon$ in the RA algorithm convergence at the assumed level of the relative convergence tolerance $\epsilon$ is examined in the ARA algorithm. Owing to these modifications the searched bounds are obtained much faster. The performed numerical simulations indicate high accuracy of algorithms RA and ARA, although the VaR estimate interval obtained in such a way is mostly very wide. It results from the fact of the unspecified dependence structure and random vector $(L_1, L_2, \ldots, L_n)$.

At the end we mention the Extended Rearrangement Algorithm (ERA) which was proposed by Bernard et al. (2017), at an additional constraint that a bound on the variance of $L$
is known, that is \( D^2(L) \leq s^2 \). As it can be expected this assumption improves the VaR estimate of bounds comparing to algorithms RA and ARA. In the ERA algorithm the authors use the risk measures of TVaR (Tail Value at Risk) and LTVaR (Left Tail Value at Risk), which are defined analogically to ES and they are sometimes called so. In this algorithm for fixed \( k \), we divide the distribution random variable \( L \) into two parts corresponding to the fixed confidence level \( 1-\alpha \), determining the matrix \( X_{N \times n} \) in such a way. Then, we use the RA algorithm to the upper and lower part of the distribution respectively, obtaining the matrix \( X^* \) of dimension \( N \times n \). When inequality \( D^2(L) \leq s^2 \) is satisfied we obtain the lower and upper VaR\( _{1-\alpha}(L) \) bounds. On basis of the ERA algorithm the authors prove that the models used so far can underestimate VaR. Moreover, they state that establishing the capital requirements at the high confidence level at e.g. 99.5% is justified.

3 The standard method of assessment of the SCR for the non-life premium and reserve risk

In the standard approach of Solvency II, the overall solvency capital requirement for insurer is calculated with use of the following formula (Solvency II Directive, 2009, Art. 103):

\[
SCR = BSCR + Adj + SCR_{op},
\]

where: \( BSCR \) - basic solvency capital requirement, \( Adj \) - adjustment for the risk absorbing effect of technical provisions and deferred taxes, \( SCR_{op} \) - the capital requirement for operational risk.

The \( BSCR \) value is determined in the aggregation process of SCRs established for the main risk modules (namely market risk, counterparty default risk, life underwriting risk, non-life underwriting risk, health underwriting risk, intangible assets risk), SCRs for the modules are determined by aggregation of SCRs for the sub-modules, and the last ones as a result of the SCR aggregation for risk carriers. Therefore, this process includes 3 levels of aggregation, they are presented in details in (Commission Delegated Regulation, 2015, CHAPTER V). The solvency capital requirements for each risk (that is the module, the sub-module and the carrier) at the specific aggregation level should correspond to the economic capital (EC) established for one year at the confidence level of 0.995 (Solvency II Directive, 2009, Note 64, s. 7). Thus, according to the definition ECs (Lelyveld, 2006) should be equal:

\[
SCR^{(l)}_{i} = VaR_{0.995}(L^{(l)}_{i}), \quad L^{(l)}_{i} = X^{(l)}_{i} - E(X^{(l)}_{i}),
\]

where \( X^{(l)}_{i} \) means the random variable which models losses, over an annual time horizon, associated with the \( i \)-risk at \( l \)-level of the aggregation (thus, \( L^{(l)}_{i} \) means unexpected losses).
In the further part of our paper we concentrate on the non-life premium and reserve risk sub-module\(^3\). The standard formula for SCR for this sub-module is following (Commission Delegated..., Art. 115):

\[
SCR_{nl\text{premium}res} = 3 \cdot \sigma_{nl} \cdot V_{nl}
\]

where: \(\sigma_{nl}\) - the standard deviation for non-life premium and reserve risk determined in accordance with (Commission Delegated Regulation, 2015, Art. 117); \(V_{nl}\) - the volume measure for non-life premium and reserve risk determined in accordance with (Commission Delegated Regulation, 2015, Art. 116):

\[
V_{nl} = \sum_s V_s,
\]

\(V_s\) - the volume measure for the premium and reserve risk of segment \(s\) (a list of the segments is provided in the (Commission Delegated Regulation, 2015, Annex II).

The key role in estimating SCR on basis of formula (2) is played by the standard deviation \(\sigma_{nl}\), which is given as (Commission Delegated Regulation, 2015, Art. 117, Para.1):

\[
\sigma_{nl} = \frac{1}{V_{nl}} \cdot \sqrt{\sum_{s,t} Corr_{S,s,t} \cdot \sigma_s \cdot V_{s} \cdot \sigma_t \cdot V_t}
\]

where: \(Corr_{S,s,t}\) - the correlation parameter for the non-life premium and reserve risk for segment \(s\) and segment \(t\) set out in (Commission Delegated Regulation, 2015, Annex IV); \(\sigma_s, \sigma_t\) - standard deviations for the non-life premium and reserve risk of segments \(s\) and \(t\) respectively; \(V_s, V_t\) - volume measures for the premium and reserve risk of segments \(s\) and \(t\), referred to in (Commission Delegated Regulation, 2015, Art. 116), respectively.

For all segments set out in Annex II, the standard deviation for the non-life premium and reserve risk shall be equal to the following (Commission Delegated Regulation, 2015, Art. 117, Para.2):

\[
\sigma_s = \frac{\sqrt{\sigma^2_{(\text{prem},s)} \cdot V^2_{(\text{prem},s)} + \sigma^2_{(\text{prem},s)} \cdot V_{(\text{prem},s)} \cdot \sigma_{(\text{res},s)} \cdot V_{(\text{res},s)} + \sigma^2_{(\text{res},s)} \cdot V^2_{(\text{res},s)}}}{V_{(\text{prem},s)} + V_{(\text{res},s)}}
\]

where: \(\sigma_{(\text{prem},s)}\) - the standard deviation for the non-life premium risk of segment \(s\) determined in accordance with (Commission Delegated Regulation, 2015, Art. 117, Para. 3); \(\sigma_{(\text{res},s)}\) - the standard deviation for the non-life reserve risk of segment \(s\) as set out in (Commission Delegated Regulation, 2015, Annex II); \(V_{(\text{prem},s)}\) - the volume measure for the premium risk of segment \(s\) referred to in (Commission Delegated Regulation, 2015, Art. 116);

\(^3\) As a result of SCR aggregation for this sub-module with SCRs for the non-life catastrophe risk sub-module and the non-life lapse risk sub-module SCR for non-life underwriting risk module is obtained. This is the second aggregation level.
$V_{(res,s)}$ - denotes the volume measure for the reserve risk of segment $s$ referred to in (Commission Delegated Regulation, 2015, Art. 116).

It results from the above that in the standard formula on $SCR_{nl\ prem\ res}$ there are not any explicit hypotheses for the distribution of the random variable that describe unexpected losses. This variable will be still marked by $L_{nl}$ (to keep transparency of the entry we omit the superscript denoting the aggregation level). However, considering the general principle saying that SCR for the specific risk should secure unexpected losses at 0.995 confidence level (cf. formula (2)) and a form of formula (3), such an assumption is needed. In particular, for a normal distribution it holds that: $VaR_{0.995}(X) = c \cdot \sigma_X$, where $c = \Phi^{-1}(0.995) = 2.58$. In the standard formula constant $c = 3$. It means that SCR is estimated at 0.9987 confidence level, thus at a higher level than generally accepted. According to legislators it is to prevent SCR underestimate when $L_{nl}$ does not have a normal distribution.

Another doubt connected with applying the standard formula on a $SCR_{nl\ prem\ res}$ concerns the dependence structure between the non-life premium and reserve risks in the particular segments. From a way of determining the parameter for the aggregated non-life premium and reserve risk (formula (5)) it results that it is described only by Pearson correlation coefficients. Since the same correlation coefficients may correspond to different dependence structures obvious questions arise. To what extent is SCR estimated in such a way reliable? How much can it differ from SCR estimated taking the proper dependence structure into account? In order to answer these questions for a specific case a study was performed and its results are presented in the next chapter.

4 SCR for the non-life premium and reserve risk - empirical results
This chapter presents the results of the study in which SCR sensitivity for the premium and reserve risk for two segments of non-life insurance and reinsurance on the dependence structure between these segments was analyzed. The segments in question are:

- motor vehicle liability insurance and proportional reinsurance (segment $s$).
- other motor insurance and proportional reinsurance (segment $t$).

The same values of the volume measures were assumed for the premium and reserve risk of segments $s$ and $t$, whereas the values of the necessary standard deviations and correlation parameter were taken as in the (Commission Delegated Regulation, 2015, Annex II and Annex IV) (see Table 1).
Then, according to the standard formula (cf. formula (3)) SCR was calculated, obtaining values $SCR_{nl\text{ prem res}} = 0.8656$ (cf. Table 2). Next it has been assumed that random variables $L_s, L_t$, which model the unexpected losses for the non-life premium and reserve risk of segments $s$ and $t$ respectively, have normal distributions: $L_s \sim N(0, V_s \cdot \sigma_s), L_t \sim N(0, V_t \cdot \sigma_t)$. For the assumed parameters (Table 1) the following results were obtained (after performing necessary calculations using formulas (3) and (6)): $L_s \sim N(0, 0.1802), L_t \sim N(0, 0.1526)$. At those assumptions, using formula (2) for variable $L_{nl} = L_s + L_t$ SCR was determined for the following cases:

- Case A: Variables $L_s$ and $L_t$ are independent.
- Case B: Variables $L_s$ and $L_t$ are comonotonic.
- Case C: Unknown dependence structure between $L_s$ and $L_t$. The unconstrained SCR upper bound was determined using TVAR, in accordance with theorem 1 in (Bernard et al., 2017).
- Case D: Unknown dependence structure. The unconstrained SCR upper bound was determined using the ARA algorithm.
- Case E: The partially known dependence structure, namely it is only known that the correlation coefficient between $L_s$ and $L_t$ is 0.5 (cf. Table 1). The constrained SCR upper bound was determined in accordance with theorem 5 in (Bernard et al., 2017).

The obtained results are presented in Table 2. The following conclusions can be drawn on basis of the research made:

- In the process of determining the solvency capital requirements for the non-life premium and reserve risk, knowledge about only the distribution of random variables $L_s$ and $L_t$ without knowledge about the dependence structure between them is not

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**Table 1. Parameters used in the study.**

<table>
<thead>
<tr>
<th>Segment $s$</th>
<th>Segment $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{(prem,s)} = 1.0$</td>
<td>$V_{(prem,t)} = 1.0$</td>
</tr>
<tr>
<td>$V_{(res,s)} = 1.2$</td>
<td>$V_{(res,t)} = 1.2$</td>
</tr>
<tr>
<td>$\sigma_{(prem,s)} = 0.10$</td>
<td>$\sigma_{(prem,t)} = 0.08$</td>
</tr>
<tr>
<td>$\sigma_{(res,s)} = 0.09$</td>
<td>$\sigma_{(res,t)} = 0.08$</td>
</tr>
</tbody>
</table>

$CorrS_{s,t} = 0.5$
sufficient. The interval of possible values for SCR from 0.6060 to 0.9342 (case D) obtained in this case is useless from the practical point of view. The same applies to case C, where the length of the interval is even greater due to the method used.

- Using only the correlation coefficient to describe the dependence structure (as it is in the Solvency II standard formula) is insufficient. The same correlation coefficient between $L_s$ and $L_t$ may correspond to different dependence structures, and thus different SCR values. In the performed study (case E) the upper bound of SCR was obtained at 0.9251 level. It means that SCR can be higher than one determined by the standard formula about 6.9%.

- By the standard formula higher SCR was obtained than at the assumption of the co-monotonic dependence structure between $L_s$ and $L_t$ (cf. case B), namely higher than the sum of SCRs for segments s and t. It means that using the standard formula on SCR for the non-life premium and reserve risk a diversification effect is not considered.

### Table 2. Research results.

<table>
<thead>
<tr>
<th>Cases</th>
<th>Capital requirement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solvency II standard formula</td>
<td>0.8656</td>
</tr>
<tr>
<td>Case A</td>
<td>0.6060</td>
</tr>
<tr>
<td>Case B</td>
<td>0.8573</td>
</tr>
<tr>
<td>Case C</td>
<td>(0.6060*, 0.9625)</td>
</tr>
<tr>
<td>Case D</td>
<td>(0.6060*, 0.9342)</td>
</tr>
<tr>
<td>Case E</td>
<td>(0.6060*, 0.9251)</td>
</tr>
</tbody>
</table>

*The lowest SCR for independent random variables $L_s$ and $L_t$ has been assumed.

### Conclusions

The paper demonstrates, using a specific example, that the use of the standard formula in accordance with the Committee Delegated Regulation (EU) 2015/35 included in Article 115 may lead to the improper level of solvency capital requirements for non-life premium and reserve risk. The conducted analysis shows that the correct estimation of SCRs depends on the correct identification of the structure of dependencies between random variables modeling unexpected losses. The application of only linear correlation coefficients for this purpose may

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4 The lowest value was SCR for independent variables $L_s$ and $L_p$. 
lead to erroneous results, since they describe only linear dependencies, in a straightforward manner. In the general case, different dependency structures may have the same value of this coefficient. The work assumes normal distribution for unexpected losses. In practice, however, these are very often asymmetrical distributions. The next stage of the research will be the analysis of the effect of skewness on the SCR evaluation.

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References


